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Analyticity and Polynomial Approximation in Modular Function Spaces

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Let E be a compact of C^N and let L_ρ be a modular functions space. In this paper we consider the problem of analytic extension of $f \in L_\rho$ to a holomorphic function defined on an open neighborhood of E. In particular, we generalize some of the results obtained in [W. Pleśniak, "Quasianalyticity in F-spaces of Integrable Functions, Approximation, and Function Spaces" (Z. Ciesielski, Ed.), pp. 553–571, Proceedings of the International Conference held in Gdańsk, August 27–31, PWN Warszawa, North-Holland, Amsterdam, 1979; W. Pleśniak, "Leja's Type Polynomial Condition and Polynomial Approximation in Orlicz Spaces," *Ann. Polon. Math.* **46** (1985), 268–278] for the case of Orlicz Spaces and in [W. M. Kozlowski and G. Lewicki, "On Polynomial Approximation in Modular Function Spaces" (J. Musielak, Ed.), pp. 63–68, Proceedings of the International Conference "Function Spaces" held in Poznań, August 1986, Teubner, Stuttgart, 1988] for the case of s-convex function modulars.

INTRODUCTION

In this paper we consider the problem of analytic extension of measurable functions. The idea of expressing some extension properties by means of polynomial approximation has its origin in S. N. Bernstein's result from 1911: if a continuous function $f: [0, 1] \to \mathbb{C}$ can be extended to a holomorphic function $\tilde{f}: U \to \mathbb{C}$ such that U is an open neighborhood (in $\mathbb{C}$) of $[0, 1]$ and $\tilde{f}(x) = f(x)$ for $x \in [0, 1]$, then $\limsup_{k \to \infty} [\operatorname{dist}_{\|\cdot\|}(f, P_k)]^{1/k} < 1$. The symbol $\operatorname{dist}_{\|\cdot\|}(f, P_k)$ denotes the supremum-norm distance between the function f and the class of all polynomials of degree not greater than k. The above idea was then developed and generalized to the case of compact subsets of $\mathbb{C}^n$ by J. Siciak [14, 15]. The first attempt to obtain similar results for measurable functions and distances different from

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the supremum norm was done by W. Pleśniak in 1985, who found some conditions for Orlicz space L^{φ} (φ was assumed to be a Δ_2-function). In this paper we consider a large class of modular function spaces which contains Orlicz spaces. In general, we prove some results without any convexity or Δ_2-assumptions; roughly speaking, this new approach consists in dealing with a distance induced by a modular instead of that induced by a norm or F-norm. This is much more convenient for applications since norms and F-norms in modular spaces are not defined in a direct way.

Since the proofs exploit many ideas of both approximation theory and modular space theory, we present a list of necessary definitions in the preliminary Sections 1 and 2. Basic theorems on analytic extension have been stated in Section 3 while in Section 4 we discuss some special cases and examples. In Section 5 we deal with quasi-analytic functions in the sense of Bernstein.

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We will need the following definition.

DEFINITION 1.1 (See, e.g., [11, 13]). Let E be a Borel subset of the space $\mathbb{C}^N$ and let μ be a Borel measure on E. We say that the pair (E, μ) satisfies the L^*-condition (the L^*-condition at a point $a \in \bar{E}$) if and only if for every family $\mathscr{F}$ of polynomials such that $\mu(t \in E\colon \sup_{p \in \mathscr{F}} |p(t)| = +\infty\}$ $= 0$ and for every $b > 1$ there exist $M > 0$ and U an open neighborhood of $\bar{E}$ (open neighborhood of a) such that

$$\sup_U |p| \leqslant M b^{\deg p} \qquad \text{for every } p \in \mathscr{F}.$$

It is clear that if E is a compact subset of $\mathbb{C}^N$ then the pair (E, μ) satisfies the L^*-condition if and only if the pair (E, μ) satisfies the L^*-condition at each point $a \in E$.

Now we will give some examples (for a more complete list of examples see [13]). By the famous polynomial lemma of Leja [6] we have the following.

EXAMPLE 1.2. If E is a rectifiable Jordan arc in $\mathbb{C}$ and μ is a length measure over E then (E, μ) satisfies L^* at every point $a \in E$.

By Fubini's theorem, from Example 1.2 we get

EXAMPLE 1.3. Let E be a subset of the space $\mathbb{R}^n$ (R^n is treated as a subset of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$). Let m_n denote the Lebesgue n-dimensional measure. (E, m_n) satisfies L^* at $a \in E$ if there exists a nonsingular affine mapping $l: I_n \to \mathbb{R}^n$ such that $a \in l(I_n) \subset E \cup \{a\}$, where I_n is the nth Cartesian power of $I = [0, 1]$. In particular, for every bounded convex set $E \subset \mathbb{R}^n$ such that int $E \neq \varnothing$ or else for every bounded Lipschitz domain (of class Lip 1) the pair (E, m_n) satisfies L^* at every point $a \in \bar{E}$. We add that the condition L^* is invariant under nondegenerate holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ (see [9, 13]).

We also have the following geometrical criterion for L^* (see [12, 13]).

THEOREM 1.4. *Given $a \in \bar{E}$, suppose that there exists an analytic mapping $h: [0, 1] \to \bar{E}$ such that $h(0) = a$. The pair (E, μ) satisfies L^* at $a \in \bar{E}$ if the pair (E, μ) satisfies L^* at $h(t)$ for each $t \in (0, 1]$.*

In Sections 3 and 5 we also need the following

DEFINITION 1.5. Let E be a compact subset of $\mathbb{C}^n$. Put $\hat{E} = \{t \in \mathbb{C}^n: |p(t)| \leqslant \|p\|_E$ for every polynomial $p\}$ ($\|p\|_E$ denotes the supremum norm on the set E). We say that E is polynomially convex if $\hat{E} = E$.

In the case when E is a compact subset of $\mathbb{C}$, by the Runge theorem, we have: E is polynomially convex if and only if $\mathbb{C} \cup \{\infty\} \setminus E$ is connected.

If $E \subset \mathbb{C}^n$ is compact, polynomially convex set the following theorem is true. $H(E)$ denotes the class of all complex-valued functions on E that can be extended to holomorphic functions in a neighborhood at E.

THEOREM 1.6 (See [14, 15]). *If $f \in H(E)$ then* $\limsup_{k \to \infty} [\mathrm{dist}_{\|\cdot\|_E}(f, P_k)]^{1/k} < 1$, *where P_k denotes the class of all polynomials of degree $\leqslant k$.*

2

Let us recall some basic concepts of the theory of modular spaces after [8]. Let X be a real or complex vector space; a functional $\rho: X \to [0, +\infty]$ is called a modular, if there holds for arbitrary $x, y \in X$:

1. $\rho(0) = 0$,
2. $\rho(\alpha x) = \rho(x)$ for every $\alpha \in K$ ($K = \mathbb{R}$ or $K = \mathbb{C}$) such that $|\alpha| = 1$,
3. $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ *for* $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

If in place of 3 there holds for some $s \in (0, 1]$

3′. $\rho(\alpha x + \beta y) \leqslant \alpha^s \rho(x) + \beta^s \rho(y)$ for $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$, then the modular ρ is called s-convex; 1-convex modular is called convex. If ρ is a modular in X then $X_\rho = \{x \in X: \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$is called a modular space; X_ρ is a vector subspace of X. For a modular ρ in X we may define the F-norm $|\cdot|_\rho$ by the formula

$$|x|_\rho = \inf\{u > 0 : \rho(x/u) \leqslant u\}.$$

If ρ is an s-convex modular then the functional

$$|x|_\rho^s = \inf\{u > 0 : \rho(x/u^{1/s}) \leqslant 1\}$$

is an s-homogeneous norm in X_ρ (a norm for $s = 1$) called the Luxemburg norm. We note the following basic properties of the above-introduced notions.

THEOREM 2.1 (See [8, Th. 1.5, Th. 1.6]). (a) *If* $\rho(\lambda x_1) \leqslant \rho(\lambda x_2)$ *for every* $\lambda > 0$, *where* $x_1, x_2 \in X_\rho$, *then* $|x_1|_\rho \leqslant |x_2|_\rho$.

(b) *If* $x \in X_\rho$, *then* $|\alpha x|_\rho$ *is a nondecreasing function of* $\alpha \geqslant 0$.

(c) *If* $|x|_\rho < 1$ *then* $\rho(x) \leqslant |x|_\rho$.

(d) If $x_n, x \in X_\rho$, *then* $|x_n - x|_\rho \to 0$ *if and only if* $\rho(\alpha(x_n - x)) \to 0$ *for every* $\alpha > 0$.

(e) *If* ρ *is s-convex, then properties* (a), (b), (c) *remain valid if we replace* $|x|_\rho$ *by* $|x|_\rho^s$.

DEFINITION 2.2 (See [8, Def. 1.7]). A modular ρ is called

(a) Right-continuous, if $\lim_{\lambda \to 1^+} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$,

(b) Left-continuous, if $\lim_{\lambda \to 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$,

(c) Continuous, if it is both right- and left-continuous.

It is easy to prove the following result (for s-convex modulars the result is given in [8, Th. 1.8].

THEOREM 2.3. (a) *If* ρ *is right-continuous then the inequalities* $|x|_\rho < 1$ *and* $\rho(x) < 1$ *are equivalent.*

(b) *If* ρ *is left-continuous then the inequalities* $|x|_\rho \leqslant 1$ *and* $\rho(x) \leqslant 1$ *are equivalent.*

We will need the following extension of the above theorem

PROPOSITION 2.4. *If* ρ *is left-continuous then the inequality* $|x|_\rho \leqslant u$ *implies*

$$\rho(x/u) \leqslant u.$$

The proof is straightforward.

DEFINITION 2.5 (see [8, Def. 5.1]). A sequence (x_k) of elements of X_ρ is called modular convergent to $x \in X_\rho$ (ρ-convergent) if there exists a $\lambda > 0$ such that $\rho(\lambda(x_k - x)) \to 0$ as $k \to \infty$.

PROPOSITION 2.6 (See [8, Prop. 5.2]). *ρ-convergence in X_ρ follows from F-norm convergence in X_ρ.*

In the sequel we shall deal with a particular class of modular spaces, modular function spaces introduced in [3, 4]. Some examples of modular function spaces will be listed below. Now we would like to give some definitions and facts of this theory; for a more complete and more general exposition of the theory the reader is referred to [3, 4].

Let E be a compact subset of $\mathbb{C}^n$ (n is a fixed natural number). By Σ we will denote the σ-algebra of subsets of E induced by the σ-algebra of all Lebesgue measurable subsets of $\mathbb{C}^n$. By $M(E)$ we will denote the space of all Σ-measurable, $\mathbb{C}$-valued functions defined on E; let S denote the subspace of all Σ-measurable simple functions, If $A \subset E$ then 1_A will stand for its characteristic function.

DEFINITION 2.7. A set function $\mu\colon \Sigma \to [0, \infty]$ will be called a σ-subadditive measure or simply a σ-submeasure if and only if

1. $\eta(\varnothing) = 0$,
2. $\eta(\bigcup_{n=1}^\infty A_n) \leqslant \sum_{n=1}^\infty \eta(A_n)$ for any sequence of $A_n \in \Sigma$,
3. $\eta(A) \leqslant \eta(B)$ if $A, B \in \Sigma$ and $A \subset B$.

DEFINITION 2.8 (See [3, 4]). A functional $\rho\colon M(E) \times \Sigma \to [0, \infty]$ is called a function modular if and only if the following conditions are satisfied:

A_1. $\rho(0, A) = 0$ for each $A \in \Sigma$;

A_2. If $A \in \Sigma$, $f, g \in M(E)$, and $|f(x)| \leqslant |g(x)|$ for all $x \in A$ then $\rho(f, A) \leqslant \rho(g, A)$;

A_3. For every $f \in M(E)$, $\rho(f, \cdot)\colon \Sigma \to [0, \infty]$ is a σ-submeasure;

A_4. For every $A \in \Sigma$ $\rho(\alpha, A) \to 0$ as $\alpha \to 0^+$, where for the sake of simplicity we write $\rho(\alpha, A)$ instead of $\rho(\alpha 1_A, A)$ ($\rho(\alpha)$ stands for $\rho(\alpha, E)$);

A_5. If $\rho(\alpha, A) = 0$ for an $\alpha > 0$ ($A \in \Sigma$) then $\rho(\beta, A) = 0$ for every $\beta > 0$;

A_6. $\rho(\alpha, \cdot)$ is order condtinuous on Σ for all $\alpha > 0$, i.e., $\rho(\alpha, A_n) \to 0$ if $A_n \to \varnothing$;

A_7. $\rho(f, A) = \sup\{\rho(g, A)\colon g \in S,\ |g(x)| \leqslant |f(x)|$ for each $x \in A\}$.

DEFINITION 2.9. A set $A \in \Sigma$ is said to be ρ-null if and only if

$\rho(\alpha, A)=0$ for every $\alpha>0$. We say that $f=g$ ρ-almost everywhere (ρ-a.e.) if the set $\{x \in E: f(x) \neq g(x)\}$ is ρ-null.

PROPOSITION 2.10. *If $A \in \Sigma$, $f \in M(E)$ then $\rho(f, A)=0$ if and only if $A \cap \operatorname{supp}(f)$ is ρ-null, where* $\operatorname{supp}(f)=\{x \in E: f(x) \neq 0\}$.

Let us put $\rho(f)=\rho(f, E)$ for every $f \in M(E)$. By the above proposition we may, identifying in $M(E)$ functions which differ only on ρ-null sets, regard a functional $\rho: M(E) \rightarrow[0, \infty]$ as a modular. Namely we have the following theorem.

THEOREM 2.11 (see [3, 4]). *The functional $\rho: M(E) \rightarrow[0, \infty]$ defined by $\rho(f)=\rho(f, E)$ is a modular.*

According to the general modular theory we can define a modular space

$$L_\rho=\{f \in M(E): \rho(\lambda f) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+\}.$$

We shall equip L_ρ with an F-norm $|\cdot|_\rho$ (s-norm $|\cdot|_\rho^s$ (s-norm $|\cdot|_\rho^s$ in the case of s-convex ρ) induced by the modular ρ. It is evident in view of A_4 that L_ρ contains all bounded measurable functions.

We note two very important results about modular function spaces.

THEOREM 2.12 (See [3, Prop. 2.6, Th. 3.6]). (a) *If f_n, $f \in M(E)$, $E \in \Sigma$, and $f_n \rightrightarrows f$ in A then $\rho(\alpha(f_n-f), A) \rightarrow 0$ for all $\alpha>0$, i.e., $|(f_n-f) 1_A|_\rho \rightarrow 0$.*

(b) *L_ρ is complete, i.e., L_ρ is an F-space (Banach space in the convex case).*

EXAMPLE 2.13. Let μ be a nonnegative finite measure defined on Σ. Let us consider a function $\varphi: E \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following conditions:

(i) For every $x \in E$ $\varphi(x, \cdot)$ is a nondecreasing, continuous function such that $\varphi(x, 0)=0$, $\varphi(x, u)>0$ for $u>0$,

(ii) $\varphi(\cdot, u)$ is a Σ-measurable, locally integrable function for all $u \geqslant 0$.

It is easily seen that $\rho(f, A)=\int_A \varphi(x,|f(x)|)\, d\mu$ is a function modular. The modular space introduced by ρ is called the Orlicz–Musielak space L^{φ} (see [2, 8]). If $\varphi(x, u)=\varphi(u)$ is independent of the variable x we say that L^{φ} is an Orlicz space.

EXAMPLE 2.14. Let M be a family of countably additive nonnegative measures on Σ and let φ be a function satisfying (i) and (ii) from Example 2.13. The modular $I_{\varphi}(f)=\sup_{\mu \in M} \int_E \varphi(x,|f(x)|)\, d\mu$ may be regarded as a function modular (see [1, 3, 4]).

EXAMPLE 2.15. In [3] it was shown that spaces of functions which are integrable with respect to linear or non-linear operator-valued measures are the modular function spaces.

Recall that a function $f \in M(E)$ is said to have an absolutely continuous F-norm if and only if for every sequence $A_n \in \Sigma$ such that $A_n \downarrow \varnothing$ there holds $|f 1_{A_n}|_\rho \to 0$. In general not all members of L_ρ have this property, therefore we shall distinguish the class of functions having it and denote it by E_ρ. Clearly E_ρ is a linear subspace of L_ρ; it plays a similar role as the so-called space of finite elements in the theory of Orlicz spaces. The most important properties of E_ρ are given in the following theorem.

THEOREM 2.16 (See [3, Ths. 4.2, 4.3, 4.5, 4.6]). (a) *E_ρ is a closure of the space of simple functions S.*

(b) (Vitali theorem) *If $f_n \in E_\rho$, $f \in L_\rho$, and $f_n \to f$ ρ—a.e. then the following conditions are equivalent*:

(i) *$f \in E_\rho$ and $|f_n - f|_\rho \to 0$,*

(ii) *For every $\alpha > 0$ $\rho(\alpha f_n, \cdot)$ are order equicontinuous, i.e., if $A_k \in \Sigma$, $A_k \downarrow \varnothing$ then $\sup_n \rho(\alpha f_n, A_k) \to 0$ as $k \to \infty$.*

(c) (The Lebesgue dominated convergence theorem) *If $f_n \to f$ ρ—a.e. ($f_n, f \in M(E)$) and there exists a function $g \in E_\rho$ such that $|f_n(x)| \leqslant |g(x)|$ ρ—a.e. for every natural n then $|f_n - f|_\rho \to 0$.*

DEFINITION 2.17. By L_ρ^0 we shall mean a class of $f \in L_\rho$ such that $\rho(f, \cdot)$ is order continuous. The smallest linear subspace of L_ρ which contains L_ρ^0 will denoted by L_ρ^c.

Now we shall give the following

DEFINITION 2.18. We say that ρ satisfies the Δ_2-condition if and only if for each sequence of $f_n \in L_\rho^c$ the following implication holds: $\rho(f_n, \cdot)$ are order equicontinuous implies $\rho(2f_n, \cdot)$ are order equicontinuous.

The above-introduced condition generalizes the Δ_2-condition used in the theory of Orlicz–Musielak spaces. We have the following characterization of modular function spaces L_ρ with ρ satisfying the Δ_2-property.

THEOREM 2.19 (See [4, Th. 3.1.6]). *If L_ρ^0 is absorbing in L_ρ (which implies $L_\rho^c = L_\rho$) then the following conditions are equivalent*:

(a) *ρ satisfies the Δ_2-condition,*

(b) *L_ρ^0 is a linear subspace of L_ρ,*

(c) $E_\rho = L_\rho^0 = L_\rho$,

(d) *The modular convergence is equivalent to the F-norm convergence in* L_ρ.

Our concept of the Δ_2-condition, though useful and structural, cannot be applied in many situations when some numerical calculations are needed. This is why we shall pose another definition.

DEFINITION 2.20. We say that the function modular ρ satisfies the Δ_2^1-condition if to every $d > 0$ there corresponds a positive number $c(d)$ such that $\rho(f+g) \leqslant c(d)$ whenever $\rho(f) \leqslant d$ and $\rho(g) \leqslant d$. Let $B(d) = \{f \in L_\rho : \rho(f) \leqslant d\}$.

THEOREM 2.21 (See [8, Th. 6.2]). *If ρ is a Δ_2-modular then there exists a $d > 0$ such that*

$$\sup\{\rho(f+g) : f,\, g \in B(d)\} = c(d) < +\infty.$$

This partial result, however, does not answer the question when both conditions Δ_2 and Δ_2^1 are equivalent. It is clear that Δ_2^1 implies Δ_2; the inverse implication, however, may not hold (see Example 2.21.a). Nevertheless, for a large class of modular function spaces, Orlicz and Orlicz-Musielak spaces included, both conditions are equivalent.

EXAMPLE 2.21.a. Let $X = [0, 1)$ and let $(X_p)_{p \in \mathbb{N}}$ be a countable disjoint partition of X such that $m(X_p) = 2^{-p}$, where m denotes the Lebesgue measure on $[0, 1)$. Let $\mathscr{P}$ be a δ-ring generated by all sets of the form $A \cap X_p$ ($p \in \mathbb{N}$, A are measurable). For a measurable function $f: X \to \mathbb{R}$ and $E \subset [0, 1)$ measurable we define

$$\rho(f, E) = \sum_{p=1}^{\infty} \left(\int_{X_p \cap E} |f|^p \, dm \right)^{1/p} + \sup \left\{ \int_{X_p \cap E} |f|^p \, dm ; p \in \mathbb{N} \right\}.$$

It is easy to verify that ρ is a function modular. Moreover, ρ does not satisfy the Δ_2^1-condition. Indeed, put $u_p = 2 \cdot 1_{X_p}$ for $p \in \mathbb{N}$. We observe that $\rho(u_p) = 2$ while

$$\rho(2u_p) \geqslant \int_{X_p} 2^p \cdot 2^p \, dm = 2^p \to \infty \qquad \text{as} \quad p \to \infty.$$

One can prove, however, that $L_\rho = L_\rho^0$; i.e., ρ satisfies the Δ_2-condition.

In the sequel the following technical rest will be frequently used.

LEMMA 2.22. *Let ρ be a function modular and let $f_k \in L_\rho$. If there exists a constant $c > 1$ such that*

$$\rho(c^k f_k) \to 0 \left(\textit{resp.} \sum_{k=1}^{\infty} \rho(c^k f_k) < +\infty, \limsup_{k \to \infty} [\rho(c^k f_k)]^{1/k} < 1 \right),$$

then for every $b \in [1, c)$

$$|b^k f_k|_\rho \to 0 \left(\textit{resp.} \sum_{k=1}^{\infty} |b^k f_k|_\rho < +\infty, \limsup_{k \to \infty} [|b^k f_k|_\rho]^{1/k} < 1 \right).$$

Proof. Fix $b \in (1, c)$ and put $d_k = \max\{(b/c)^k, \rho(c^k f_k)\}$. Then $\rho(b^k f_k / d_k) \leqslant \rho(c^k f_k) \leqslant d_k$ which gives $|b^k f_k|_\rho \leqslant d_k$. The rest of the proof is elementary. In the case of s-convex ρ we can obtain a similar result for $|\cdot|_\rho^s$.

3

Applying Theorem 1.6 we may prove the following

THEOREM 3.1. *Let E be a compact, polynomially convex subset of $\mathbb{C}^n$ and let ρ be a function modular. Denote $H_\rho(E) = H(E) \cap L_\rho$. There exists a constant $c > 1$ such that $\lim_{k \to \infty} \operatorname{dist}_\rho(c^k f, P_k) = 0$, where P_k denotes the class of all polynomials of degree $\leqslant k$ and*

$$\operatorname{dist}_\rho(c^k f, P_k) = \inf\{\rho(c^k f - p) : p \in P_k\} \qquad \textit{for} \quad k \in \mathbb{N}.$$

Proof. By Theorem 1.6 for every function f, holomorphic in a neighborhood of E, there holds $\limsup_{k \to \infty} [\operatorname{dist}_{\|\ \|_E}(f, P_k)]^{1/k} < 1$. This means that there exists a constant $b < 1$ such that $\|f - p_k\|_E \leqslant b^k$ for some $p_k \in P_k$ and $k \geqslant k_0$. Let $c > 1$ be such that $c \cdot b < 1$. By A_2 and A_4 of Definition 2.8 we get

$$\begin{aligned} \operatorname{dist}_\rho(c^k f, P_k) &\leqslant \rho(c^k(f - p_k)) \leqslant \rho(\|c^k(f - p_k)\|_E) \leqslant \rho(c^k \|f - p_k\|_E) \\ &\leqslant \rho(c^k \cdot b^k) \leqslant \rho((cb)^k) \to 0 \qquad \text{as} \quad k \to \infty. \end{aligned}$$

This completes the proof.

By similar reasoning and using Remark 5.2 in [10] one can easily obtain the following result.

Remark 3.2. Let E be a compact subset of $\mathbb{C}^n$ and left f be an entire function. Then for every $c > 1$, $\lim_{k \to \infty} \operatorname{dist}_\rho(c^k f, P_k) = 0$.

Remark 3.3. Assume that ρ, E are such as in Theorem 3.1. Assume furthermore that for every $a \in (0, 1)$ $\limsup_{k\to\infty}[\rho(a^k, E)]^{1/k} < 1$. If $f \in H_\rho(E)$ then $\limsup_{k\to\infty}[\mathrm{dist}_{|\cdot|_\rho}(f, P_k)]^{1/k} < 1$.

Proof. Using the estimation from Theorem 3.1 we have

$$\rho(c^k(f - p_k)) \leqslant \rho(a^k, E) \qquad \text{where} \quad a = c \cdot b < 1.$$

By the assumption on ρ and by Lemma 2.22 we have $\limsup_{k\to\infty}[|f - p_k|_\rho]^{1/k} < 1$ which gives the result.

The next theorem gives a partially converse result to Theorem 3.1.

THEOREM 3.4. *Let ρ be a function modular. Let E be a compact subset of $\mathbb{C}^n$ and let $\mu \ll \rho$ (i.e., μ equals zero on ρ-null sets) be a Borel measure such that (E, μ) satisfies L^*. If for a function $f \in L_\rho$ we may choose a constant $d > 1$ and an increasing sequence $k_i \in \mathbb{N}$ such that $k_i \to +\infty$, $\limsup_{i\to\infty} k_{i+1}/k_i < \infty$, and $\sum_{i=1}^\infty \mathrm{dist}_\rho(d^{k_i}f, P_{k_i}) < \infty$ then $f \in H_\rho(E)$.*

Proof. By Lemma 2.22 we can choose $c \in (1, d)$ such that $\sum_{i=1}^\infty \mathrm{dist}_{|\cdot|_\rho}(c^{k_i}f, P_{k_i}) < \infty$. Put $d_j = \mathrm{dist}_{|\cdot|_\rho}(c^{k_j}f, P_{k_j}) + 2^{-j}$ for $j = 1, 2, \ldots$. Then to every $j \in \mathbb{N}$ there corresponds a polynomial $p_j \in P_{k_j}$ such that $|c^{k_j}(f - p_j)|_\rho < d_j$. We note that

$$\begin{aligned}|c^{k_j}(p_{j+1} - p_j)|_\rho &\leqslant |c^{k_j}(f - p_{j+1})|_\rho + |c^{k_j}(f - p_j)|_\rho \leqslant |c^{k_{j+1}}(f - p_{j+1})|_\rho \\ &\quad + |c^{k_j}(f - p_j)|_\rho \leqslant d_{j+1} + d_j.\end{aligned}$$

Consequently, in view of Theorem 2.1.c, for j sufficiently large we get

$$\rho(c^{k_j}(p_{j+1} - p_j)) \leqslant |c^{k_j}(p_{j+1} - p_j)|_\rho \leqslant d_{j+1} + d_j.$$

Now define the set

$$D = \{t \in E : \sup_{j \in N} c^{k_j}\, |p_{j+1}(t) - p_j(t)| = +\infty\}.$$

We shall prove that $\mu(D) = 0$. For $j, n \in \mathbb{N}$ put

$$E_{j,n} = \{t \in E : c^{k_j}\, |p_{j+1}(t) - p_j(t)| > n\}$$

and $E_n = \bigcup_{j=n}^\infty E_{j,n}$. It is easy to see that $E_n \supset E_{n+1}$ and $\bigcap_{n=1}^\infty E_n = D$. Let us fix $\alpha > 0$; for $n \geqslant \alpha$ we have

$$\begin{aligned}\rho(\alpha, E_{j,n}) &\leqslant \rho(n, E_{j,n}) \leqslant \rho(c^{k_j}(p_{j+1} - p_j), E_{j,n}) \leqslant \rho(c^{k_j}(p_{j+1} - p_j)) \\ &\leqslant d_j + d_{j+1}.\end{aligned}$$

Thus,

$$\rho(\alpha, E_n) \leqslant \sum_{j=n}^{\infty} \rho(\alpha, E_{j,n}) \leqslant \sum_{j=n}^{\infty} d_j + \sum_{j=n}^{\infty} d_{j+1} \to 0$$

as $n \to \infty$, because $\sum_{j=1}^{\infty} d_j < \infty$.

Since $D \subset E_n$ for every $n \in N$, it follows that $\rho(\alpha, D) \leqslant \rho(\alpha, E_n) \to 0$ and consequently $\rho(\alpha, D) = 0$ for all $\alpha > 0$. The latter fact implies $\mu(D) = 0$, since $\mu \ll \rho$. Hence by the L^*-condition, for every $b > 1$ there exist $M > 0$ and an open neighborhood U of E such that

$$\sup_{t \in U} |c^{k_j}(p_{j+1}(t) - p_j(t))| \leqslant b^{k_{j+1}} \cdot M \qquad \text{for} \quad j = 1, 2, \ldots .$$

Choose $b > 1$ such that $b/c^{1/k} < 1$, where $k > \lim_j \sup k_{j+1}/k_j$. Compute

$$\sup_{t \in U} |p_{j+1}(t) - p_j(t)| \leqslant M \frac{b^{k_{j+1}}}{c^{k_j}} \leqslant M \frac{b^{k_{j+1}}}{c^{(k_j/k_{j+1}) \cdot k_{j+1}}} \leqslant M \left(\frac{b}{c^{1/k}}\right)^{k_{j+1}}$$

for j sufficiently large. This means that the series

$$p_1 + \sum_{j=1}^{\infty} (p_{j+1} - p_j)$$

is uniformly convergent in U to a holomorphic function $\tilde{f}$. This implies by Theorem 2.12a, that $|p_j 1_E - \tilde{f} 1_E|_\rho \to 0$. Hence, since $|p_j - f|_\rho \to 0$, $f = \tilde{f} 1_E$ ρ—a.e. which gives $\tilde{f} = f 1_E$ μ—a.e. by the absolute continuity of μ with respect to ρ. The proof of Theorem 3.4 is fully completed.

4

Let us consider the following condition:

(4.1) There exists $K > 1$ such that $\rho(2f) \leqslant K\rho(f)$ for every $f \in L_\rho$;

(4.2) To every $b > 1$ there corresponds $c > 1$ and $k_0 \in \mathbb{N}$ such that $\rho(c^k f) \leqslant b^k \rho(f)$ for every $f \in L_\rho$ and $k \geqslant k_0$.

LEMMA 4.3. *Conditions* (4.1) *and* (4.2) *are equivalent.*

Proof. If ρ satisfies (4.1) then its is easy to show by the induction that $\rho(2^n f) \leqslant K^n \rho(f)$ for every $f \in L_\rho$ and $n \in \mathbb{N}$. Fix $b > 1$. If $b \geqslant K$ then we can put $c = 2$ and $k_0 = 1$. So assume that $b \in (1, K)$ and choose $k_0 \in N$ such that $b^{k_0} \geqslant K$. Put $c = 2^{1/2k_0}$ and fix $k \geqslant k_0$. We can write $k = mk_0 + p$ where $m, p \in \mathbb{N}$, $m \geqslant p$, and $p < k_0$. Compute

$$b^k \rho(f) \geqslant b^{mk_0} \rho(f) \geqslant K^m \rho(f) \geqslant \rho(2^m f) = \rho(c^{2mk_0} f) \geqslant \rho(c^{mk_0 + p} f) = \rho(c^k f).$$

To prove the converse, fix an arbitrary $b>1$. By (4.2) we get constants $k_0 \in N$ and $c>1$ such that $\rho(c^k f) \leqslant b^k \rho(f)$ for $k \geqslant k_0$, $f \in L_\rho$. Choose $k_1 \geqslant k_0$ such that $c^{k_1} \geqslant 2$. Thus

$$\rho(2f) \leqslant \rho(c^{k_1} f) \leqslant b^{k_1} \rho(f),$$

which completes the proof.

Using Lemma 4.3 we can prove the following

THEOREM 4.4. *Let ρ, μ, and E be the same as in the assumptions of Theorem* 3.4. *Assume additionally that ρ satisfies* (4.1) *and that there exists a strictly increasing sequence (k_j) of natural numbers such that* $\limsup_{j \to \infty} k_{j+1}/k_j \leqslant 0$ *and*

$$\limsup_{j \to \infty} [\operatorname{dist}_\rho(f, P_{k_j})]^{1/k_j} < 1. \tag{4.5}$$

Then $f \in H_\rho(E)$.

Proof. In view of Theorem 3.4 it suffices to find a constant $c>1$ such that

$$\sum_{j=1}^{\infty} \operatorname{dist}_\rho(c^{k_j} f, P_{k_j}) < \infty.$$

From (4.5) it follows that there exists $d \in (0, 1)$ such that $\operatorname{dist}_\rho(f, P_{k_j}) \leqslant d^{k_j}$ for j sufficiently large. Choose $b>1$ such that $bd<1$. Then

$$b^{k_j} \operatorname{dist}_\rho(f, P_{k_j}) < (db)^{k_j}$$

and by Lemma 4.3 we may find $c>1$ such that

$$\operatorname{dist}_\rho(c^{k_j} f, P_{k_j}) \leqslant b^{k_j} \operatorname{dist}_\rho(f, P_{k_j}) < (bd)^{k_j}.$$

Thus the series

$$\sum_{j=1}^{\infty} \operatorname{dist}_\rho(c^{k_j} f, P_{k_j})$$

is convergent, which completes the proof.

Remark 4.4.a. Note that the F-norm $|\cdot|_\rho$ induced by an arbitrary function modular ρ may be regarded itself as a function modular. It is clear that $|\cdot|_\rho$ satisfies (4.1). Hence, we may replace (4.5) by

$$\limsup_{j \to \infty} [\operatorname{dist}_{|\cdot|_\rho}(f, P_{k_j})]^{1/k_j} < 1 \tag{4.6}$$

and Theorem 4.4 remains valid. In case of Orlicz spaces L^φ with φ satisfying the Δ_2-condition such a result was given in [13].

We would like to stress that, in general, it is much easier to work with conditions imposed on modulars than with conditions involving F-norms because of their indirect definition. However, in some cases both conditions (4.5) and (4.6) are equivalent.

PROPOSITION 4.7. *If ρ satisfies* (4.1) *then* (4.6) *is equivalent to* (4.5).

Proof. Since $\rho(f) \leqslant |f|_\rho < 1$ holds, it follows that (4.6) always implies (4.5). To prove the converse let us choose $d \in (0, 1)$ such that $\operatorname{dist}_\rho(f, P_k) < d^k$ for $k \geqslant k_0$. There exists a sequence of $w_k \in P_k$ such that $\rho(f - w_k) < d^k$. Let $b > 1$ be so chosen that $bd < 1$. By Lemma 4.3 there exists $c > 1$ and $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$

$$\rho(c^k(f - w_k)) \leqslant b^k \rho(f - w_k) \leqslant (bd)^k \to 0 \qquad \text{as} \quad k \to \infty.$$

The rest of the proof will be divided into two parts.

(a) Let ρ be s-convex, $s \in (0, 1]$. Since $\rho(c^k(f - w_k)) \leqslant 1$ for large k, then

$$|f - w_k|_\rho^s \leqslant (1/c^k)^s = (1/c^s)^k$$

and

$$[\operatorname{dist}_{|\ |_\rho^s}(f, P_k)]^{1/k} \leqslant (|f - w_k|_\rho^s)^{1/k} \leqslant 1/c^s < 1.$$

Hence,

$$\limsup_{k \to \infty} [\operatorname{dist}_{|\cdot|_\rho}(f, P_k)]^{1/k} \leqslant 1/c^s < 1.$$

(b) Suppose ρ is not s-convex. Put $q = \max(bd, 1/c)$, then we have $\rho((f - w_k)/q^k) \leqslant q^k$ and consequently $|f - w_k|_\rho \leqslant q^k$. Hence,

$$\limsup_{k \to \infty} [\operatorname{dist}_{|\ |_\rho}(f, P_k)]^{1/k} \leqslant q < 1.$$

PROPOSITION 4.8. *Assume that to every $a > 1$ there corresponds $d > 0$ such that*

$$|cf|_\rho \geqslant (1 + d) \cdot |f|_\rho \qquad \textit{for all} \quad f \in L_\rho. \tag{4.9}$$

If there exists $c > 0$ such that $\operatorname{dist}_\rho(c^k f, P_k) \to 0$ *then*

$$\limsup_{k \to \infty} [\operatorname{dist}_{|\ |_\rho}(f, P_k)]^{1/k} < 1.$$

Proof. By Lemma 2.22 we can find $b \in (1, c)$ such that $\mathrm{dist}_{|\cdot|_\rho}(b^k f, P_k) \to 0$ and consequently $|b^k(f - p_k)|_\rho \to 0$ for some $p_k \in P_k$. By (4.9) we get

$$|f - p_k|_\rho (1 + d)^k \leqslant |b^k(f - p_k)|_\rho < 1$$

for k sufficiently large.

Let us note that every s-homogeneous norm satisfies (4.9) with $1 + d = c^s$. To obtain a sufficient condition for (4.9) formulated in terms of a modular we define the following function:

$$w(t) = \sup\{\rho(tf)/\rho(f) : f \in L_\rho \setminus \{0\}\}.$$

PROPOSITION 4.10. *If to every $c > 1$ there corresponds $d > 0$ such that*

$$w\left(\frac{1+d}{c}\right) \cdot (1 + d) \leqslant 1, \tag{4.11}$$

then (4.9) *holds.*

Proof. Fix $c > 1$ and choose $d > 0$ such that (4.11) holds. Put

$$L = (1 + d)^{-1} \cdot \{\alpha > 0 : \rho(cf/\alpha) \leqslant \alpha\}$$

and

$$P = \{\beta > 0 : \rho(f/\beta) \leqslant \beta\}.$$

We claim that $L \subset P$. Indeed, let $\gamma \in L$; hence,

$$\begin{aligned} \rho(f/\lambda) = \rho\left(\frac{1+d}{c} \cdot \frac{cf}{\gamma(1+d)}\right) &\leqslant w\left(\frac{1+d}{c}\right) \rho\left(\frac{cf}{\gamma(1+d)}\right) \\ &\leqslant w\left(\frac{1+d}{c}\right)(1+d)\gamma \leqslant \gamma, \qquad \text{i.e., } \gamma \in P. \end{aligned}$$

We have $(1 + d)^{-1} |cf|_\rho = \inf L \geqslant \inf P = |f|_\rho$ which gives (4.9).

Using Theorem 4.4, Remark 4.4a, and Propositions 4.8 and 4.10 we obtain immediately the following result which generalizes Theorem 2.3 of [5].

THEOREM 4.12. *Let ρ, μ, and E be the same as in the assumptions of Theorem* 3.4. *Let $f \in L_\rho$ and let us assume additionally that ρ satisfies* (4.11) (*in particular ρ can be s-convex or it can satisfy* (4.9)). *If* $\lim_{k\to\infty} \mathrm{dist}_\rho(c^k f, P_k) = 0$ *then $f \in H_\rho(E)$.*

Now we will show a possible method of application of the above.

EXAMPLE 4.13. Let $E \subset \mathbb{C}$ be a sphere of radius $r \in (0, 1)$ and center at zero. Consider a function $f \in L_\rho$ defined by the formula

$$f(x) = \sum_{k=0}^{\infty} a_k f_k(x),$$

where $a_k \in \mathbb{C}$, f_k are measurable functions defined in $\mathbb{C}$, and the convergence of the series is understood in the sense of $|\cdot|_\rho$. Assume $I_k \subset E$ are measurable, $f_k(x) = x^k$ for every $x \in I_k$, and $|a_k| \leqslant M$ for all $k \in \mathbb{N}$. We would like to find out when f may be extended to a holomorphic function in a neighborhood of E. Let μ be the length measure in E and let ρ be a function modular satisfying (4.11) such that μ is absolutely continuous with respect to ρ. By Example 1.2 (E, μ) satisfies the L^*-condition in this case. In view of Theorem 4.12 it suffices to find $d > 1$ such that $\operatorname{dist}_\rho(d^k f, P_k) \to 0$ as $k \to \infty$. Denoting

$$w_k = \sum_{i=0}^{k} a_i x^i \qquad \text{and} \qquad D_k = E \backslash I_k,$$

we get

$$\begin{aligned}\operatorname{dist}_\rho(d^k f, P_k) &\leqslant \rho(d^k(f - w_k)) \leqslant \rho(d^k(f - w_k), I_k) + \rho(d^k(f - w_k), D_k)\\ &\leqslant \rho\left(d^k \sum_{i=k+1}^{\infty} |a_i|\, r^i\right) + \rho(d^k(f - w_k), D_k)\\ &\leqslant \rho(Mr(1-r)^{-1}(dr)^k) + \rho(d^k(f - w_k), D_k).\end{aligned}$$

The first term on the right tends to 0 for $dr < 1$. The question of analytic extension of f has been, therefore, reduced to the question whether $\rho(d^k(f - w_k), D_k)$ tends to zero. The latter problem may be solved in various ways depending on the form of the modular. For instance, let $\rho(f) = \int_E \varphi(|f|)\, d\mu$ be an Orlicz modular with φ satisfying the Δ_2-condition. Observe that if there exists a function $g \in L_\rho$ such that $|f(x) - w_k(x)| \leqslant |g(x)|$ for $x \in D_k$ and sufficiently large $k \in \mathbb{N}$, then

$$\begin{aligned}\rho(d^k(f - w_k), D_k) &= \int_{D_k} \varphi(d^k |f - w_k|)\, d\mu \leqslant \tilde{M}^k \int_{D_k} \varphi(|f - w_k|)\, d\mu\\ &\leqslant \tilde{M}^k \int_{D_k} \varphi(|g|)\, d\mu,\end{aligned}$$

where $\tilde{M}$ is the Δ_2-constant. We conclude then, that in order to prove $f \in H_\rho(E)$ it suffices to check whether $\int_{D_k} \varphi(|g|)\, d\mu$ tends to zero faster than $\tilde{M}^k$ tends to infinity.

5

Let ρ be a function modular and $Z \subset L_\rho$; define $R(Z) = \inf\{r(f): f \in Z \setminus \{0\}\}$, where $r(f) = \sup\{|tf|_\rho : t \geqslant 0\}$. Similarly we may put $R_\rho(Z) = \inf\{r_\rho(f) : f \in Z \setminus \{0\}\}$, where $r_\rho(f) = \sup\{\rho(tf) : t \geqslant 0\}$.

For an arbitrary function modular ρ we have

Proposition 5.1. *$R(L_\rho) = 0$ if and if $R_\rho(L_\rho) = 0$.*

Proof. Assume $R(L_\rho) = 0$; fix $\varepsilon \in (0, 1)$, and choose a function $f \in L_\rho \setminus \{0\}$ such that $r(f) < \varepsilon$. Then we have

$$r_\rho(f) = \sup\{\rho(tf) : t \geqslant 0\} \leqslant \sup\{|tf|_\rho : t \geqslant 0\} = r(f) < \varepsilon$$

and by the arbitrariness of ε we get $R_\rho(L_\rho) = 0$. Let $R_\rho(L_\rho) = 0$; to every $\varepsilon > 0$ there corresponds a function $f \in L_\rho \setminus \{0\}$ such that $\rho((t/\varepsilon)f) < \varepsilon$ for every $t \geqslant 0$. Hence, $|tf|_\rho < \varepsilon$ for arbitrary $t \geqslant 0$ and consequently $R(L_\rho) = 0$.

In the sequel we will need the following two definitions.

Definition 5.2. A non-ρ-null set $A \in \Sigma$ is called an atom if and only for every $B \in \Sigma$, $B \subset A$ there holds either B is ρ-null or $\rho(\alpha, B) = \rho(\alpha, A)$ for all $\alpha > 0$.

Definition 5.3. We say that a function modular ρ is atomless in $E_1 (E_1 \subset E,\ E_1 \in \Sigma,\ E_1$ is not ρ-null) if and only if there is no atom which is included in E_1.

Let us consider the following conditions:

(C_1) If $A_n \in \Sigma$ and $A_n \downarrow \varnothing$ then $\lim_{n \to \infty}(\sup\{\rho(\alpha, A_n) : \alpha \geqslant 0\}) = 0$,

(C_2) There exists a constant $d > 0$ such that $\sup\{\rho(\alpha, A) : \alpha \geqslant 0\} \geqslant d$ for every $A \in \Sigma$ which is not ρ-null.

Theorem 5.9 will state that, under some assumptions on ρ, there exist in L_ρ some quasi-analytic functions (cf. Def. 5.7) that cannot be extended to holomorphic functions. In the proof of that result we apply the "lethargy" theorem of Bernstein for F-spaces (Th. 5.6). To apply this theorem we have to assume that $R(Z) > 0$ form some nontrivial subsets Z of L_ρ. It follows immediately from the definition that $R(Z) = \infty$ for $Z \neq \{0\}$ and s-convex modulars ($0 \leqslant s \leqslant 1$). Condition ($C_1$), which implies $R(L_\rho) = 0$, is, for instance, satisfied for the modular

$$\rho(f) = \int_0^\infty \phi(f(x))\, dx,$$

where $\phi(0)=0$, ϕ is nondecreasing even, nonnegative, and $\lim_{u\to\infty}\phi(u)<\infty$. $R(Z)>0$ whenever (C_2) holds. The latter is, for example, satisfied by Orlicz modulars for which

$$\lim_{u\to\infty}\phi(u)=\infty.$$

PROPOSITION 5.4. *If ρ satisfies (C_1) and there is a set $E_1\subset E$ such that ρ is atomless in E_1 then $R_\rho(L_\rho)=0$.*

Proof. Since ρ is atomless in E_1 we can choose a sequence (A_n) of Σ-measurable subsets of E_1 such that $A_n\downarrow\varnothing$ and A_n is not ρ-null for $n=1, 2, \ldots$. Put $V_n=\{f\in S: f1_{E\setminus A_n}=0\}$. Observe that V_n are nontrivial subspaces of L_ρ. For every $f\in V_n$ there holds

$$r_\rho(f)=\sup\{\rho(tf): t\geqslant 0\}=\sup\{\rho(tf, A_n): t\geqslant 0\}\leqslant\sup\{\rho(\alpha, A_n): \alpha\geqslant 0\}.$$

Hence,

$$\begin{aligned}R_\rho(L_\rho)&=\inf\{r_\rho(f): f\in L_\rho\setminus\{0\}\}\\&\leqslant\inf\left\{r_\rho(f): f\in\bigcup_{n=1}^{\infty}V_n\setminus\{0\}\right\}\leqslant\sup\{\rho(\alpha, A_n): \alpha\geqslant 0\}\end{aligned}$$

for $n=1, 2, \ldots$ and by (C_1) $R_\rho(L_\rho)=0$.

PROPOSITION 5.5. *If ρ satisfies (C_2) then $R_\rho(Z)>0$ for every nontrivial subset Z of L_ρ.*

Proof. Let $f\in L_\rho\setminus\{0\}$, denote $A_n=\{x\in E: |f(x)|\geqslant 1/n\}$. Note that (A_n) is nondecreasing and A_n are not ρ-null for $n\geqslant n_0$. Then for $n\geqslant n_0$ we have

$$r_\rho(f)\geqslant\rho(n^2f)\geqslant\rho(n^2f, A_n)\geqslant\rho(n, A_n)\geqslant\rho(n, A_{n_0}).$$

Hence,

$$r_\rho(f)\geqslant\sup\{\rho(\alpha, A_{n_0}): \alpha\geqslant 0\}\geqslant d$$

and consequently $R_\rho(L_\rho)\geqslant d>0$.

Let us present the following version of Bernstein's "lethargy" theorem.

THEOREM 5.6 (see [7]). *Let $(Y, |\cdot|)$ be an F-space and let $V_1\subsetneqq V_2\subsetneqq\cdots\subsetneqq Y$ be a nested sequence of distinct, finite-dimensional vector subspaces of Y.*

Assume that

$$R\left(\bigcup_{n=1}^{\infty} V_n\right) > 0.$$

Then for every decreasing sequence (d_n) *of nonnegative real numbers with* $\lim_n d_n = 0$ *there exists* $y \in Y$ *such that* $\mathrm{dist}_{|\cdot|_\rho}(y, V_n) = d_n$ *for* n *sufficiently large.*

We shall use the above theorem to generalize Theorem 5.2.2 from [13] to the case of modular function spaces.

DEFINITION 5.7 (See [11, Def. 4.1]). Given a Borel subset E of the space $\mathbb{C}^n$, a function $f \in L_\rho$ is said to be quasi-analytic on E (in the sense of Bernstein) if there exists an increasing sequence (k_j) of positive integers and polynomials p_j with $\deg p_j \leqslant k_j$ $(j = 1, 2, ...)$ such that

$$\limsup_{j \to \infty} [|f - p_j|_\rho]^{1/k_j} < 1. \tag{5.8}$$

It is interesting to study when a quasi-analytic function belongs to $H_\rho(E)$. By Remark 4.4.a this holds for (E, μ) satisfying L^* and $\limsup_{j\to\infty} k_{j+1}/k_j < \infty$. In the next theorem it will be shown that this assumption on the sequence (k_j) in Remark 4.4.a is essential.

THEOREM 5.9. *If* E *is a polynomially convex, compact subset of* $\mathbb{C}^n$,

$$\limsup_{j\to\infty} k_{j+1}/k_j = \infty, \qquad R\left(\bigcup_{k=1}^{\infty} P_k\right) > 0,$$

and

$$\limsup_{k\to\infty} [\rho(a^k)]^{1/k} < 1 \qquad \textit{for every} \quad a \in (0, 1)$$

then there exists a quasi-analytic function $f \in L_\rho \setminus H_\rho(E)$.

Proof. We note that by assumption on the sequence (k_j) there is a subsequence (k_{j_m}) of (k_j) such that $\lim_{m\to\infty} k_{j_{m+1}}/k_{j_m} = \infty$. For simplicity we leave (k_j) to be the subsequence. Fix $a \in (0, 1)$. By Theorem 5.6 there exists a function $f \in L_\rho$ such that for $k \geqslant k_0$, $\mathrm{dist}_{|\cdot|_\rho}(f, P_k) = d_k$, where $d_k = a^{k_j}$ as $k_j \leqslant k < k_{j+1}$, for $j = 1, 2, ...$. We note that

$$\lim_{j\to\infty} [\mathrm{dist}_{|\cdot|_\rho}(f, P_{k_j})]^{1/k_j} = \lim_{j\to\infty} (a^{k_j})^{1/k_j} = a < 1.$$

On the other hand,

$$\limsup_{k\to\infty} [\operatorname{dist}_{|\cdot|_\rho}(f, P_k)]^{1/k} \geqslant \limsup_{j\to\infty} [\operatorname{dist}_{|\cdot|_\rho}(f, P_{k_{j+1}-1})]^{1/k_{j+1}-1}$$
$$\geqslant \lim_{j\to\infty} a^{k_j/k_{j+1}-1} = 1.$$

By Remark 3.3 f does not belong to $H_\rho(E)$.

Let us denote the set of all quasi-analytic functions by $B_\rho(E)$. It is known that, if the set of all polynomials is dense in L_ρ, then $L_\rho = B_\rho(E) + B_\rho(E)$ (see [11, Prop. 1.4]). This can be surprising in comparison with the following strong identity principle.

THEOREM 5.10 (See [11] in the case of Orlicz spaces). *Assume that E is a connected open set in $\mathbb{R}^n$, ρ a function modular, and let μ be a Borel measure on E, $\mu \ll \rho$. Assume furthermore that for each compact interval $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ contained in E the pair (I, μ) satisfies L^*. Let $f \in B_\rho(E)$. If $f = 0$ on a subset F of E such that (F, μ) satisfies L^* then $f = 0$ μ—a.e. on E.*

Proof. At first suppose that E is a compact interval included in $\mathbb{R}^n$. Since $f \in B_\rho(E)$ there exists a strictly increasing sequence of integers k_j, polynomials p_j with $\deg p_j \leqslant k_j$ and a constant $a \in (0, 1)$ such that $|f - p_j|_\rho \leqslant a^{k_j}$ for j sufficiently large. We note that

$$|p_j 1_F|_\rho = |(f - p_j)\, 1_F|_\rho \leqslant |f - p_j|_\rho$$

which gives

$$|p_j 1_F|_\rho \leqslant a^{k_j} \qquad \text{for } j \text{ sufficiently large.}$$

Choose $b > 1$ such that $ab < 1$. Since any F-norm satisfies condition (4.1), by Lemma 4.3, there exists $c \in (1, b)$ such that

$$|c^{k_j} p_j 1_F|_\rho \leqslant b^{k_j} |p_j 1_F|_\rho < (ab)^{k_j}. \tag{5.11}$$

Let $D = \{t \in F\colon \sup_{j\in N} |c^{k_j} p_j(t)| = +\infty\}$. We claim that $\mu(D) = 0$. To prove this define for $j, n \in \mathbb{N}$

$$F_{n,j} = \{t \in F\colon |c^{k_j} p_j(t)| > n\} \qquad \text{and} \qquad F_n = \bigcup_{j=n}^{\infty} F_{n,j}.$$

Using (5.11), by a similar reasoning as in Theorem 3.4, we can show that for every $\alpha > 0$, $\lim_{n\to\infty} \rho(\alpha, F_n) = 0$ which gives $\rho(\alpha, D) = 0$ and finally

$\mu(D)=0$. Now choose $d \in (1, c)$. By condition L^* there exists a closed subinterval I_0 of E and a constant $M>0$ such that

$$\sup_{t \in I_0} |p_j(t)| \leqslant M(d/c)^{k_j} \qquad \text{for } j \text{ sufficiently large.}$$

Hence, the sequence p_j tends uniformly to 0 in I_0 which gives $f=0$ μ—a.e. on I_0. Let J_0 be a maximal element of the family $\mathscr{J}$ of all compact subintervals I of E such that $I_0 \subset I$ and $f=0$ μ—a.e. on I. We claim that $J_0=E$. Indeed, since $f=0$ μ—a.e. on J_0 we have $|p_j 1_{J_0}| \leqslant a^{k_j}$ for j sufficiently large and since (J_0, μ) satisfies L^* we can again choose a compact interval J such that $J_0 \subset \operatorname{int} J$ and p_j tend uniformly to 0 in J as $j \to \infty$. Thus $f=0$ μ—a.e. on $J \subset E$, hence $J \cap E = J_0$. This is, however, possible only if $J_0=E$ as claimed.

Thus, we can prove our theorem in the case when E is a compact interval. The rest of the proof proceeds along similar lines as in [11, Lemma 5.1, Corollary 5.2, and Theorem 5.4] so we omit details.

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